

MAS212 Linear Algebra I 2007-2008

Coursework 8 — Solutions

Exercise* 1.

(a) L is linear, since, if $A, B \in \mathbb{R}^{n \times n}$ and $\alpha, \beta \in \mathbb{R}$, then

$$L(\alpha A + \beta B) = 3(\alpha A + \beta B) = 3\alpha A + 3\beta B = \alpha L(A) + \beta L(B).$$

(b) L is linear, since, if $A, B \in \mathbb{R}^{n \times n}$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} L(\alpha A + \beta B) &= (\alpha A + \beta B)^T + (\alpha A + \beta B) = \alpha A^T + \beta B^T + \alpha A + \beta B \\ &= \alpha(A^T + A) + \beta(B^T + B) = \alpha L(A) + \beta L(B). \end{aligned}$$

(c) L is not linear since $L(O) = -I \neq O$.

(d) L is linear, since, if $A, B \in \mathbb{R}^{n \times n}$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} L(\alpha A + \beta B) &= C(\alpha A + \beta B) - (\alpha A + \beta B)C = \alpha CA + \beta CB - \alpha AC - \beta BC \\ &= \alpha(CA - AC) + \beta(CB - BC) = \alpha L(A) + \beta L(B). \end{aligned}$$

(e) L is linear, since, if $A, B \in \mathbb{R}^{n \times n}$ and $\alpha, \beta \in \mathbb{R}$, then

$$L(\alpha A + \beta B) = C^2(\alpha A + \beta B) = \alpha C^2 A + \beta C^2 B = \alpha L(A) + \beta L(B).$$

(f) We need to distinguish two cases: if $C = O$, then L is clearly linear. On the other hand, if $C \neq O$, then L is not linear, since, for example,

$$L(2I) = (2I)^2 C = 4C \neq 2C = 2I^2 C = 2L(I).$$

Exercise* 2.

(a) L is not linear, since $(L(\mathbf{0}))(t) = t \neq \mathbf{0}(t)$, so it does not map the zero polynomial to the zero polynomial.

(b) L is linear, since if $\mathbf{p}, \mathbf{q} \in P_2$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} (L(\alpha \mathbf{p} + \beta \mathbf{q}))(t) &= t(\alpha \mathbf{p} + \beta \mathbf{q})'(t) + (\alpha \mathbf{p} + \beta \mathbf{q})''(t) = \alpha t\mathbf{p}'(t) + \beta t\mathbf{q}'(t) + \alpha \mathbf{p}''(t) + \beta \mathbf{q}''(t) \\ &= \alpha(t\mathbf{p}'(t) + \mathbf{p}''(t)) + \beta(t\mathbf{q}'(t) + \mathbf{q}''(t)) = \alpha(L(\mathbf{p}))(t) + \beta(L(\mathbf{q}))(t), \end{aligned}$$

so

$$L(\alpha \mathbf{p} + \beta \mathbf{q}) = \alpha L(\mathbf{p}) + \beta L(\mathbf{q}).$$

Exercise* 3. Suppose that L is one-to-one. We need to show that if $L(\mathbf{v}) = \mathbf{0}$, then $\mathbf{v} = \mathbf{0}$. In order to see this, suppose that $L(\mathbf{v}) = \mathbf{0}$. Then $L(\mathbf{v}) = \mathbf{0} = L(\mathbf{0})$, so $\mathbf{v} = \mathbf{0}$ because L is one-to-one.

Conversely, suppose that $\ker(L) = \{\mathbf{0}\}$. We need to show that if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ then $\mathbf{v}_1 = \mathbf{v}_2$. In order to see this, suppose that $L(\mathbf{v}_1) = L(\mathbf{v}_2)$. But then $L(\mathbf{v}_1 - \mathbf{v}_2) = L(\mathbf{v}_1) - L(\mathbf{v}_2) = \mathbf{0}$, hence $\mathbf{v}_1 - \mathbf{v}_2 \in \ker(L)$, and thus $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$, that is $\mathbf{v}_1 = \mathbf{v}_2$, since $\ker(L) = \{\mathbf{0}\}$.

Exercise* 4. (a) Since

$$L(\mathbf{e}_1) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad L(\mathbf{e}_2) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

the matrix representation of L with respect to the standard bases is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(b) Since

$$L(\mathbf{e}_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad L(\mathbf{e}_2) = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}, \quad L(\mathbf{e}_3) = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix},$$

the matrix representation of L with respect to the standard bases is

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -4 & 1 \end{pmatrix}.$$

(c) Since

$$L(\mathbf{e}_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad L(\mathbf{e}_2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad L(\mathbf{e}_3) = \begin{pmatrix} 1 \\ 7 \end{pmatrix},$$

the matrix representation of L with respect to the standard bases is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 7 \end{pmatrix}.$$

Exercise* 5.

(a) Let $\mathcal{E}$ be the standard basis for $\mathbb{R}^3$. Since

$$L(\mathbf{e}_1) = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \quad L(\mathbf{e}_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad L(\mathbf{e}_3) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

the matrix representation of L with respect to the standard bases is

$$[L]_{\mathcal{E}}^{\mathcal{E}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

(b) Observe that $[L]_{\mathcal{B}}^{\mathcal{B}}$ and $[L]_{\mathcal{E}}^{\mathcal{E}}$ are related by

$$[L]_{\mathcal{B}}^{\mathcal{B}} = P_{\mathcal{B}}^{-1} [L]_{\mathcal{E}}^{\mathcal{E}} P_{\mathcal{B}}$$

where $P_{\mathcal{B}}$ is the transition matrix from $\mathcal{B}$ to the standard basis given by

$$P_{\mathcal{B}} = \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 0 \\ -4 & 12 & -7 \end{pmatrix}.$$

But, by Exercise 3 (b) on Coursework 7,

$$P_{\mathcal{B}}^{-1} = \begin{pmatrix} -7 & 3 & -2 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix},$$

so

$$[L]_{\mathcal{B}}^{\mathcal{B}} = P_{\mathcal{B}}^{-1} [L]_{\mathcal{E}}^{\mathcal{E}} P_{\mathcal{B}} = \begin{pmatrix} -7 & 3 & -2 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 0 \\ -4 & 12 & -7 \end{pmatrix} = \begin{pmatrix} -11 & 26 & -21 \\ -5 & 15 & -9 \\ -2 & 10 & -3 \end{pmatrix}.$$

Exercise* 6. (a) In order to calculate the matrix representation of L with respect to $\mathcal{P}$, we need to calculate $L(\mathbf{p}_j)$ for $j = 1, 2, 3$ and express each result as a linear combination of the basis vectors in $\mathcal{P}$. Now

$$\begin{aligned} (L(\mathbf{p}_1))(t) &= t\mathbf{p}'_1(t) + \mathbf{p}''_1(t) = t \cdot 0 + 0 = 0, \\ (L(\mathbf{p}_2))(t) &= t\mathbf{p}'_2(t) + \mathbf{p}''_2(t) = t \cdot 1 + 0 = t, \\ (L(\mathbf{p}_3))(t) &= t\mathbf{p}'_3(t) + \mathbf{p}''_3(t) = t \cdot 2t + 2 = 2 + 2t^2, \end{aligned}$$

so

$$\begin{aligned} L(\mathbf{p}_1) &= 0\mathbf{p}_1 + 0\mathbf{p}_2 + 0\mathbf{p}_3 \\ L(\mathbf{p}_2) &= 0\mathbf{p}_1 + 1\mathbf{p}_2 + 0\mathbf{p}_3 \\ L(\mathbf{p}_3) &= 2\mathbf{p}_1 + 0\mathbf{p}_2 + 2\mathbf{p}_3 \end{aligned}$$

hence

$$[L]_{\mathcal{P}}^{\mathcal{P}} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(b) In order to find the transition matrix from $\mathcal{Q}$ to $\mathcal{P}$ we need to express the vectors $\mathbf{q}_j$ for $j = 1, 2, 3$ as linear combinations of the basis vectors in $\mathcal{P}$. Now

$$\begin{aligned} \mathbf{q}_1 &= 1\mathbf{p}_1 + 0\mathbf{p}_2 + 0\mathbf{p}_3 \\ \mathbf{q}_2 &= 0\mathbf{p}_1 + 1\mathbf{p}_2 + 0\mathbf{p}_3 \\ \mathbf{q}_3 &= 1\mathbf{p}_1 + 0\mathbf{p}_2 + 1\mathbf{p}_3 \end{aligned}$$

so

$$S = [Id]_{\mathcal{P}}^{\mathcal{Q}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As the transition matrix from $\mathcal{P}$ to $\mathcal{Q}$ is just the inverse of S , we find (after a quick Gauss-Jordan inversion)

$$S^{-1} = [Id]_{\mathcal{Q}}^{\mathcal{P}} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) Since

$$[L]_{\mathcal{Q}}^{\mathcal{Q}} = [Id]_{\mathcal{Q}}^{\mathcal{P}} [L]_{\mathcal{P}}^{\mathcal{P}} [Id]_{\mathcal{P}}^{\mathcal{Q}} = S^{-1} [L]_{\mathcal{P}}^{\mathcal{P}} S,$$

we obtain

$$[L]_{\mathcal{Q}}^{\mathcal{Q}} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(d) Let $A = [L]_{\mathcal{Q}}^{\mathcal{Q}}$. First we notice that the Composition Formula implies that $[L^n]_{\mathcal{Q}}^{\mathcal{Q}} = A^n$, so by the Matrix Representation Theorem,

$$[L^n(\mathbf{p})]_{\mathcal{Q}} = [L^n]_{\mathcal{Q}}^{\mathcal{Q}} [\mathbf{p}]_{\mathcal{Q}} = A^n [\mathbf{p}]_{\mathcal{Q}}.$$

But A is diagonal, so

$$A^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}.$$

Thus, since $[\mathbf{p}]_{\mathcal{Q}} = (\alpha_1, \alpha_2, \alpha_3)^T$, we find

$$[L^n(\mathbf{p})]_{\mathcal{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_2 \\ 2^n \alpha_3 \end{pmatrix},$$

and hence

$$L^n(\mathbf{p}) = \alpha_2 \mathbf{q}_2 + 2^n \alpha_3 \mathbf{q}_3, \quad \text{or} \quad (L^n(\mathbf{p}))(t) = \alpha_2 t + 2^n \alpha_3 (1 + t^2).$$